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Detail Fatigue Analysis Exceedance Spectra Determined from Random Multidirectional Force Components

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Prediction of fatigue life of structural details currently relies on semiempirical methods based on test data from specimens undergoing cyclic loading in one direction. In actual service many structural details undergo cyclic loading from random externally applied forces in many directions. Components of internal loads in a structural detail resulting from random externally applied loading are also random and are generally statistically correlated. Procedures are provided for determining the loading exceedence spectrum in a single critical direction for a specific structural detail undergoing random loading from many directions. These procedures include both time domain and power spectral density applications and both direct and covariance treatments.

Nomenclature

$\bar{A}_l$	= rms value of $l(i\omega)$ for unit σ_w
b_1, b_2	= rms values of σ_w for the time spent in nonstorm and storm turbulence, respectively
$[C_f]$	= covariance matrix of external force components
$[C_\phi]$	= covariance matrix of external force component velocities
$\{f\}$	= vector of external force components
i	= square root of -1
l	= detail internal load
$N(x)$	= number of cumulative exceedances of level x
N_{0_l}	= number of excursions of l per unit time with positive slope through the steady-state value of l
P_1, P_2	= fractions of time spent in nonstorm and storm turbulence, respectively
$u_x, u_y, u_z, \ldots$	= detail internal load influence coefficients with respect to force components in $x, y, z, \ldots$ directions
$x, y, z, \ldots$	= components of external forces
x_0	= mean value of x
$\langle \boldsymbol{y}(t), \boldsymbol{z}(t) \rangle$	= vector with components $y(t)$ and $z(t)$
$\alpha, \beta, \gamma, \ldots$	= components of external force velocities for $x, y, z, \ldots$, respectively
λ	= velocity of l
ρ_{yz}	= correlation coefficient of y with respect to z
σ_x	= rms value of x
σ_w	= rms value of gust velocity
ω	= frequency in rad/s

Introduction

CURRENT structural detail fatigue analysis methods rely on data from test specimens undergoing cyclic loading along a single axis (usually uniaxial loading) to aid in the determination of detail fatigue life.[1] These data are in the form of stress level S vs number of cycles N to crack initiation (typically referred to as S–N data). For structural details that have principle stress axes that are closely aligned with the direction of loading, the use of the S–N data is appropriate. However, there are many structural details that undergo random loading from more than one direction. An example of this in aeronautics is the attachment support structure that couples an engine nacelle and strut with the wing. The nacelle attachment has flexibility in more than one direction and is susceptible to random multidirectional loading from flight through atmospheric turbulence and travel over rough runways. The designer is faced with the problem of how to combine the loading from many directions into a single exceedance spectrum that is compatible with the uniaxial S–N data.

Attempting to combine the loading by using vector summations will lead to single directional loading, but there is no guarantee that this resultant loading will be aligned with the detail axis, along which the fatigue damage will accumulate. In fact, it is possible that the resultant direction may be aligned perpendicular to this critical direction and result in producing no fatigue damage at all, leading to gross overestimation of the detail fatigue life. A procedure is developed to determine the exceedance spectra for structural details undergoing random multidirectional loading. This method combines statistical properties of random variables with knowledge of the detail internal load influence coefficients.

Time-Domain Applications

There are two possible approaches to determine the detail exceedance spectra, each with its own advantages. Both methods yield identical results for random multidirectional loading. At this point, a general comment is in order. Fatigue spectra are usually expressed in terms of the alternating component of a load or stress about its mean value. Throughout the remainder of this report, all references to loads or stresses will assume that the mean value has already been subtracted out and that only the alternating component about the mean is of interest, unless otherwise indicated.

Direct Method

If knowledge of structural detail internal loading influence coefficients with respect to external loading force components is known, the internal loading is simply

$$l(t) = u_x x(t) + u_y y(t) + \cdots$$
$$= \lfloor u \rfloor \{ \boldsymbol{f}(t) \} \tag{1}$$

where the external force components have been determined

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through time-domain analysis of test data time histories [when $x(t)$, $y(t)$, . . . are obtained by time-domain analysis rather than as test data, the usual procedure would be to incorporate Eq. (1) directly into the analysis as a final step]. Once the time history of $l(t)$ is known, an exceedance count can be performed by either manual or automated means[2] (usually the latter for long time history data). The advantage to the direct method is that the determination of the detail exceedance spectrum is found exactly, regardless of the loading statistical distribution; i.e., the loading does not have to be a stationary random process. However, there are several disadvantages to this method. First, knowledge of the detail influence coefficients is required prior to performing the time history analysis. Also, the computational cost of exceedance counting is currently high for longs sets of time history data. In addition, the analysis and exceedance count must be repeated if the detail influence coefficients change, which is often the case during the design process.

Covariance Method

If the loading can be considered to be a stationary Gaussian random process, an alternative to the direct method may be possible to use. Suppose that $l(t)$ is a function of two external force components, in particular, those in the y and z directions

$$l(t) = u_y y(t) + u_z z(t) \tag{2}$$

If the external loadings are random and the time history is collected for a sufficiently long period, then the graph of $z(t)$ vs $y(t)$ will fill in an elliptical region with the densest concentration of points near the origin (Fig. 1). This elliptical region is defined by the combined probability density function[3] of the random variables y and z. If the major axis of the elliptical region is not aligned with the y or z axes, then some correlation between $y(t)$ and $z(t)$ exists. This suggests that the statistical relationships of multiple random variables could be used to determine fatigue spectra, which are essentially probability assessments of the likelihood of exceeding any given load level.

Contours of constant internal load are parallel lines perpendicular to a line defined by the direction of the vector $\langle \boldsymbol{u}_y, \boldsymbol{u}_z \rangle$. Graphically it can be seen that at any time point $l(t)$ is the projection (inner product) of the vector $\langle \boldsymbol{y}(t), \boldsymbol{z}(t) \rangle$ onto the line with direction defined by the constant vector $\langle \boldsymbol{u}_y, \boldsymbol{u}_z \rangle$ (Fig. 2). The intent is to develop an exceedance spectrum of $l(t)$ as it traverses the line defined by the direction of $\langle \boldsymbol{u}_y, \boldsymbol{u}_z \rangle$. (If the forces y and z are not simple forces, but rather, for example, a vertical inertia force n_z and an aerodynamic couple M_z caused by propeller aerodynamics, the lines of l in Fig. 2 are still readily determined straight lines; the vector interpretation illustrated in the figure is not essential).

Rice[4] showed that the number of crossings (i.e., exceedances) N per unit time of a level x for a stationary Gaussian

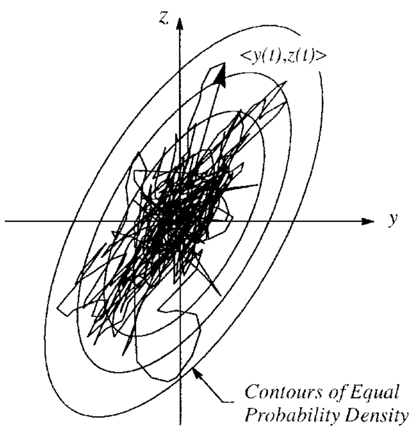

Fig. 1 ***y* vs *z* force.**

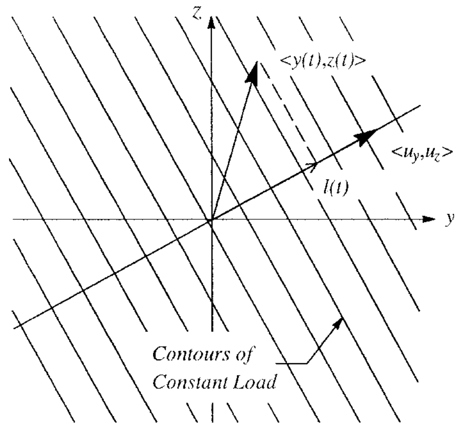

Fig. 2 Projection of $\langle y, z \rangle$ on $\langle u_y, u_z \rangle$.

random process is related to the rms level of x, that is, σ_x, and the rms velocity of x, that is, σ_α, where α is the velocity of x, in the following expression:

$$N(x) = \frac{1}{2\pi} \frac{\sigma_\alpha}{\sigma_x} \exp\left(\frac{-x^2}{2\sigma_x^2}\right) \tag{3}$$

When x is equal to zero in Eq. (3), $N(x)$ is N_0, the number of zero crossings of x with positive slope per unit time (also known as the characteristic frequency), and is equal to $(1/2\pi)(\sigma_\alpha/\sigma_x)$. Equation (3) thus gives the number of cumulative exceedances for a given level of x, and is therefore an expression for the exceedance spectrum of x.

To develop the exceedance spectrum for l, it is necessary to find σ_l and σ_λ. The inner product expression of Eq. (2) can be written as

$$l(t) = \lfloor u_y \; u_z \rfloor \begin{Bmatrix} y(t) \\ z(t) \end{Bmatrix} \tag{4}$$

The value of the square of $l(t)$ is

$$l^2(t) = \lfloor u_y \; u_z \rfloor \begin{Bmatrix} y(t) \\ z(t) \end{Bmatrix} \lfloor y(t) \; z(t) \rfloor \begin{Bmatrix} u_y \\ u_z \end{Bmatrix}$$

$$= \lfloor u_y \; u_z \rfloor \begin{bmatrix} y^2(t) & y(t)z(t) \\ z(t)y(t) & z^2(t) \end{bmatrix} \begin{Bmatrix} u_y \\ u_z \end{Bmatrix} \tag{5}$$

The mean square value of $l(t)$ is the time-averaged value of $l^2(t)$

$$\overline{l^2} = \lfloor u_y \; u_z \rfloor \begin{bmatrix} \frac{1}{N}\sum_{i=1}^{N} y_i^2 & \frac{1}{N}\sum_{i=1}^{N} y_i z_i \\ \frac{1}{N}\sum_{i=1}^{N} z_i y_i & \frac{1}{N}\sum_{i=1}^{N} z_i^2 \end{bmatrix} \begin{Bmatrix} u_y \\ u_z \end{Bmatrix} \tag{6}$$

where N is the number of sample points in the time history. The mean square value of y (with its mean subtracted out, as noted earlier) is[5]

$$\sigma_y^2 = \overline{y^2} = \frac{1}{N}\sum_{i=1}^{N} y_i^2 \tag{7}$$

The degree of correlation between y and another variable z can be found by determining the correlation coefficient,[3] that is, ρ_{yz}

$$\rho_{yz} = \frac{1}{N}\sum_{i=1}^{N} y_i z_i \bigg/ \sqrt{\frac{1}{N}\sum_{i=1}^{N} y_i^2 \frac{1}{N}\sum_{i=1}^{N} z_i^2}$$

$$= \frac{\sigma_{yz}}{\sigma_y \sigma_z} \tag{8}$$

The correlation coefficient is a measure of the phasing relationship between y and z and exists over the range -1 to 1. A value of 1 indicates that y and z are perfectly in phase, whereas a value of -1 indicates that y and z are perfectly out of phase. Intermediate values indicate the degree to which y and z are in phase with each other, with a zero value, indicates that y and z are completely uncorrelated.

Using the statistical properties of Eqs. (7) and (8), in combination with Eq. (6), gives

$$\sigma_l^2 = \overline{l^2} = \lfloor u_y \; u_z \rfloor \begin{bmatrix} \sigma_y^2 & \rho_{yz}\sigma_y\sigma_z \\ \rho_{zy}\sigma_z\sigma_y & \sigma_z^2 \end{bmatrix} \begin{Bmatrix} u_y \\ u_z \end{Bmatrix} \tag{9}$$

where the 2 by 2 matrix in Eq. (9) is called the covariance matrix of y and z. Having found the rms value of l, it now remains to find the rms value of the velocity of l. Taking the time derivative of Eq. (2) gives

$$\lambda = \frac{dl}{dt} = u_y \frac{dy}{dt} + u_z \frac{dz}{dt}$$

$$= u_y\beta + u_z\gamma \tag{10}$$

Replacing the values of y and z with their time derivatives in Eqs. (4–9) results in

$$\sigma_\lambda^2 = \overline{\left(\frac{dl}{dt}\right)^2}$$

$$= \lfloor u_y \; u_z \rfloor \begin{bmatrix} \sigma_\beta^2 & \rho_{\beta\gamma}\sigma_\beta\sigma_\gamma \\ \rho_{\gamma\beta}\sigma_\gamma\sigma_\beta & \sigma_\gamma^2 \end{bmatrix} \begin{Bmatrix} u_y \\ u_z \end{Bmatrix} \tag{11}$$

where σ_β and σ_γ are the rms values of the velocities of x and y, respectively. Now it is possible to substitute σ_λ and σ_l into Eq. (3) and determine the exceedance spectrum for l. This spectrum is in terms of the structural detail internal load and now allows for the use of S–N data for fatigue life prediction.[1]

Expansion to more than two directions of external random loading is straightforward. In general form the covariance method can be expressed as

$$\sigma_l^2 = \lfloor \boldsymbol{u} \rfloor [C_f]\{\boldsymbol{u}\} \tag{12}$$

$$\sigma_\lambda^2 = \lfloor \boldsymbol{u} \rfloor [C_\phi]\{\boldsymbol{u}\} \tag{13}$$

$$N(l) = \frac{1}{2\pi}\frac{\sigma_\lambda}{\sigma_l}\exp\left(-\frac{l^2}{2\sigma_l^2}\right) \tag{14}$$

There are several advantages to the covariance method, provided the external loading components are close to stationary Gaussian random processes. First, once the covariance matrices for the external loadings and their velocities have been found, it is possible to find the exceedance spectrum of any detail with little computational effort provided the influence coefficients are known. Second, as long as the covariance matrices are saved, it is a simple matter to determine the exceedance spectra at a later time for either additional details or details that have been redesigned and whose influence coefficients have changed. If the redesign of the detail results in a major change to the stiffness characteristics and changes the dynamic response significantly, then new covariance matrices will have to be determined.

An example of the use of the covariance method is given at the end of this article.

Continuous Turbulence Applications

Structural analysis for airplanes experiencing loading from continuous atmospheric turbulence is customarily based on power spectral density techniques. In these techniques, continuous turbulence is assumed to be a stationary Gaussian random process (support for the stationary Gaussian assumption is offered in Chap. 12 of Ref. 5). In particular, the exceedance spectrum for l may be found by the generalized exceedance expression

$$N(l) = N_{0_l}P_1\exp\left(-\frac{l/\bar{A}_l}{b_1}\right) + N_{0_l}P_2\exp\left(-\frac{l/\bar{A}_l}{b_2}\right) \tag{15}$$

where the atmospheric turbulence parameters have been determined empirically.[6] $\bar{A}_l$ and N_{0_l} must be determined to define the exceedance spectrum of l caused by continuous turbulence.

Direct Method

If the internal load influence coefficients are known before the gust analysis is performed, the direct method may be used. The expressions for $\bar{A}_l$ and N_{0_l} in the frequency domain are found by

$$\bar{A}_l = \frac{\sigma_l}{\sigma_w} = \sqrt{\int_0^\infty |l(i\omega)|^2\Phi(\omega)\,d\omega} \Bigg/ \sqrt{\int_0^\infty \Phi(\omega)\,d\omega} \tag{16}$$

$$N_{0_l} = \frac{1}{2\pi}\left[\sqrt{\int_0^\infty \omega^2|l(i\omega)|^2\Phi(\omega)\,d\omega} \Bigg/ \sqrt{\int_0^\infty |l(i\omega)|^2\Phi(\omega)\,d\omega}\right] \tag{17}$$

where $\Phi(\omega)$ is the empirically determined or specified power spectral density of atmospheric turbulence, where the value of σ_w has been normalized to a value of one.[6] $l(i\omega)$ is the frequency response function of l caused by continuous turbulence, found by

$$l(i\omega) = u_x x(i\omega) + u_y y(i\omega) + \cdots \tag{18}$$

where the frequency response functions of x, y, ... are determined using power spectral density gust analysis methods.[5] As in the use of the direct method in the time domain, if the influence coefficients change or additional details need to be analyzed, then the solution procedures for $\bar{A}_l$ and N_{0_l} need to be performed again.

Covariance Method

Fortunately, the derivation of the covariance method is not difficult in the frequency domain. If the frequency response function of y is known, then the mean square value of y is[5]

$$\sigma_y^2 = \overline{y^2} = \int_0^\infty |y(i\omega)|^2\Phi(\omega)\,d\omega \tag{19}$$

and the correlation coefficient is[5]

$$\rho_{yz} = \frac{1}{\sigma_y\sigma_z}\int_0^\infty \Phi(\omega)[y_R(\omega)z_R(\omega) + y_I(\omega)z_I(\omega)]\,d\omega \tag{20}$$

where the subscripts R and I denote the real and imaginary components of y and z. Recalling Eq. (16), Eq. (9) becomes

$$\bar{A}_l^2 = \lfloor u_y \; u_z \rfloor \begin{bmatrix} \bar{A}_y^2 & \rho_{yz}\bar{A}_y\bar{A}_z \\ \rho_{zy}\bar{A}_z\bar{A}_y & \bar{A}_z^2 \end{bmatrix} \begin{Bmatrix} u_y \\ u_z \end{Bmatrix} \tag{21}$$

The expression for N_{0_l} is

$$N_{0_l} = \frac{1}{2\pi}\frac{\sigma_\lambda}{\sigma_l} = \frac{1}{2\pi}\frac{\sigma_\lambda/\sigma_w}{\sigma_l/\sigma_w}$$

$$= \frac{1}{2\pi}\frac{\bar{A}_\lambda}{\bar{A}_l} \tag{22}$$

where

$$\bar{A}_\lambda^2 = \lfloor u_y \; u_z \rfloor \begin{bmatrix} \bar{A}_\beta^2 & \rho_{\beta\gamma}\bar{A}_\beta\bar{A}_\gamma \\ \rho_{\gamma\beta}\bar{A}_\gamma\bar{A}_\beta & \bar{A}_\gamma^2 \end{bmatrix} \begin{Bmatrix} u_y \\ u_z \end{Bmatrix} \tag{23}$$

$\bar{A}_l$ and N_{0_l} can now be substituted into Eq. (15) to determine the gust exceedance spectrum for l. Once again, it is not difficult to expand the previous derivation beyond two directions. Power spectral density gust analyses were performed using both the direct and covariance methods, and identical values for $\bar{A}_l$ and N_{0_l} were obtained from both methods for various structural details.

A word of caution for analysts using symmetric airplane models for vertical continuous turbulence analysis and antisymmetric airplane models for lateral continuous turbulence analysis: fatigue damage on certain structural components (such as nacelles and winglets) may be imposed from a combination of loading from vertical and lateral gust excitation. Fuller et al.[7] derive the following expressions to account for gust loading from all directions:

$$\bar{A}_C = \sqrt{\bar{A}_S^2 + \bar{A}_A^2} \tag{24}$$

$$N_{0_C} = \sqrt{N_{0_S}^2\bar{A}_S^2 + N_{0_A}^2\bar{A}_A^2} \Big/ \sqrt{\bar{A}_S^2 + \bar{A}_A^2} \tag{25}$$

where the subscripts S and A represent symmetric and antisymmetric quantities, respectively, and the subscript C represents combined quantities.

Example

Consider an airplane that is taxiing at relatively low speed, such that the contribution from aerodynamic forces is negligible, leaving only inertial and thrust forces acting on the nacelle. A structural detail is chosen with internal loading, a function of the vertical and lateral forces acting at the nacelle c.g. (Fig. 3).

For this example, these forces were found using accelerometer data from flight test, which was processed to compute the vertical and lateral accelerations at the nacelle c.g. A small interval of this data is shown in Fig. 4.

Using the direct method, the internal loading of this detail can be expressed as

$$\begin{aligned} l(t) &= u_y y(t) + u_z z(t) \\ &= -3.5881y(t) + 0.5054z(t) \end{aligned} \tag{26}$$

Converting the accelerations in Fig. 4 to forces using the engine mass and substituting into Eq. (26) gives the time history of the detail internal load (Fig. 5). The mean load and standard deviation of $l(t)$ were calculated to be 2335 and 4448 lb, respectively. Once the mean load has been found, it can be subtracted from $l(t)$ to give the alternating load. Now the total number of zero crossings with positive slope, N_0 times the length of record, can be determined by inspection, in this case 15 (note that for alternating load, the zero level is taken about the mean). In addition, the number of cumulative exceedances of alternating load can be counted to give the exceedance spectra for this small time interval. This is done by identifying the peaks and valleys between zero crossings and then classifying the number of peaks and valleys that fall into arbitrary bands[2] (Table 1).

Using the covariance method, the fatigue spectrum of $l(t)$ can be determined as outlined in Eqs. (12–14). First, the time histories of the engine c.g. accelerations must be converted to forces at the engine c.g. using the engine mass. The covariance

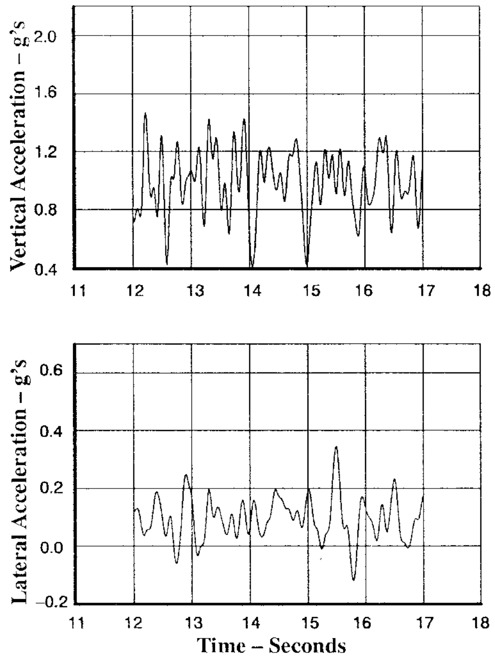

Fig. 4 Engine c.g. accelerations.

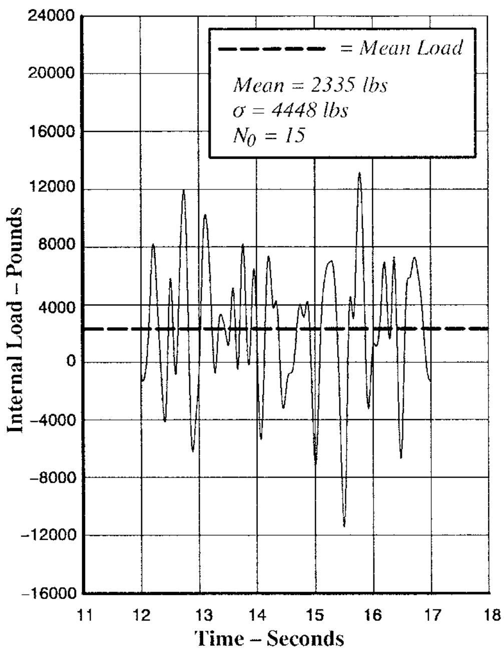

Fig. 5 Detail internal load.

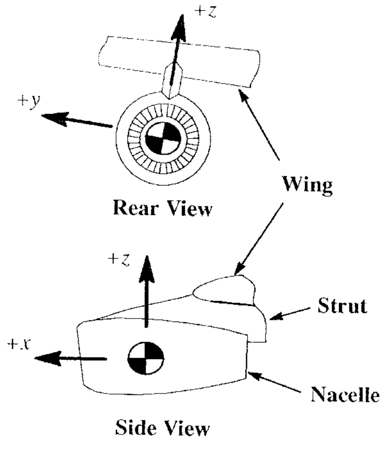

Fig. 3 Nacelle/strut installation.

Table 1 Number of exceedances of internal load

Band	Peaks	Valleys	Average	Cumulative
0–1000	0	1	0.5	15.5
1–2000	2	1	1.5	15.0
2–3000	1	2	1.5	13.5
3–4000	1	4	2.5	12.0
4–5000	4	0	2.0	9.5
5–6000	4	2	3.0	7.5
6–7000	0	1	0.5	4.5
7–8000	1	1	1.0	4.0
8–9000	0	1	0.5	3.0
9–10,000	1	2	1.5	2.5
10–11,000	1	0	0.5	1.0
11–12,000	0	0	0.0	0.5
12–13,000	0	0	0.0	0.5
13–14,000	0	1	0.5	0.5

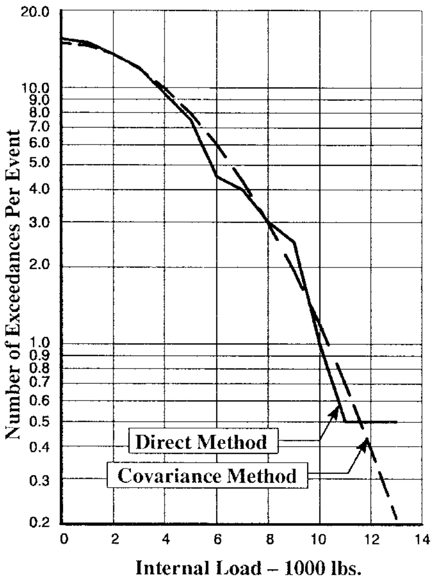

Fig. 6 Comparison of direct and covariance methods.

matrix of the engine c.g. forces is determined per Eq. (6). The rate covariance matrix is determined in the same manner by using the time history of the rates of the engine c.g. forces. For this example

$$\lfloor \boldsymbol{u} \rfloor = [-3.5881 \quad 0.5054], \quad [C_f] = \begin{bmatrix} 0.0056 & -0.0013 \\ -0.0013 & 0.0459 \end{bmatrix}$$

$$[C_\phi] = \begin{bmatrix} 1.7004 & -0.4913 \\ -0.4913 & 28.5578 \end{bmatrix}$$

$$\sigma_l = \sqrt{W^2 \lfloor u \rfloor [C_f] \{u\}} = 4448 \text{ lb}$$

$$\sigma_\lambda = \sqrt{W^2 \lfloor u \rfloor [C_\phi] \{u\}} = 83{,}474 \text{ lb/s}$$

$$N(l) = \frac{1}{2\pi} \frac{\sigma_\lambda}{\sigma_l} t \exp\left(-\frac{l^2}{2\sigma_l^2}\right)$$

$$= 14.93 \exp\left(-\frac{l^2}{39.569 \times 10^6}\right)$$

where W is the engine and strut weight of 15,000 lb and t is the time length of the data record of 5 s. The value of σ_l is the same using either method. The product of N_{0_l} and t using the direct method is 15 compared to 14.93 using the covariance method. The comparison of the methods is shown graphically in Fig. 6. In these calculations, the sample rate used for the engine accelerations was 200/s. A 15-Hz filter was used to smooth the acceleration data. The acceleration rates were found simply by using the slopes between samples of the filtered data. While this method can often lead to erratic rate data, the high sample rate and smoothing filter eliminated this problem.

Concluding Remarks

Determination of fatigue spectra for structural details undergoing complex loading from many directions is possible if certain statistical quantities of the loading can be determined and if the internal load influence coefficients are known. In the case of continuous turbulence, the phasing relationship between loads imposed from aerodynamic forces and from inertial forces can be accounted for in the fatigue spectra, and contributions from vertical and lateral turbulence can be combined properly. Test data in the form of loading time histories that exhibit near-Gaussian characteristics may be reduced to covariance matrix form, allowing for the fatigue spectra of many details to be easily determined.

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